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Generalized Fixed-Point Theorems in Complete Metric Spaces and Their Applications

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Abstract

Fixed-point theory provides a fundamental framework for establishing existence, uniqueness, and constructive approximation of solutions to nonlinear equations. This paper presents a unified treatment of generalized contractive mappings in complete metric spaces by considering a symmetric Hardy–Rogers-type condition incorporating Banach-, Kannan-, and Chatterjea-type distance terms. For a self-mapping T on a complete metric space (X, d), we study the inequality

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)],$$

where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. An explicit contraction factor for successive Picard iterates is obtained, and it is proved that every mapping satisfying the proposed condition possesses a unique fixed point. The convergence of the Picard sequence is established without imposing continuity of the mapping as a separate assumption. Quantitative a priori and a posteriori error bounds are derived, together with an Ulam–Hyers-type stability estimate for approximate fixed points. Classical contraction principles of Banach, Kannan, and Chatterjea are recovered as immediate special cases. Applications are presented to Fredholm-type nonlinear integral equations and finite-dimensional nonlinear iterative systems. A numerical illustration demonstrates geometric convergence and confirms the theoretical error behavior. The framework emphasizes the usefulness of generalized metric contractions as a common language for existence theory, iterative approximation, and stability analysis in nonlinear problems.

Keywords: Fixed point theorem, Complete metric space, Generalized contraction, Picard iteration, Hardy–Rogers contraction, Kannan mapping, stability, Integral equation

1. Introduction

Fixed-point theory is one of the central tools of nonlinear functional analysis. A fixed point of a mapping $T : X \to X$ is a point $x^* \in X$ satisfying

$$Tx^* = x^*. \tag{1}$$

Many problems originally formulated as differential equations, integral equations, nonlinear algebraic equations, optimization conditions, or iterative numerical schemes can be transformed into an equation

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of the form (1). Consequently, an appropriate fixed-point theorem can simultaneously establish the existence of a solution, its uniqueness, and a constructive procedure for approximating it. The classical starting point is Banach's contraction principle. Banach's 1922 [1] work established the foundational contraction argument in complete metric spaces and has subsequently become a standard existence-and-uniqueness mechanism.

For a complete metric space (X, d), the classical condition

$$d(Tx, Ty) \leq kd(x, y), \qquad 0 \leq k < 1, \tag{2}$$

ensures the existence of a unique fixed point and convergence of successive approximations.

The linear Lipschitz condition (2), however, is sufficient rather than necessary for fixed-point convergence. Kannan [2] introduced another contractive mechanism involving the distances of points from their own images,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \qquad k < \frac{1}{2}, \tag{3}$$

which also produces a unique fixed point in a complete metric space. The standard bibliographic record identifies Kannan's original result as *Some Results on Fixed Points*, published in 1968. Chatterjea [3] subsequently developed another form based on cross-distances between x and Ty, and between y and Tx. These developments motivated contractive conditions containing several metric quantities simultaneously. Hardy and Rogers [4] formulated an important generalization involving a linear combination of direct, displacement, and cross-distance terms. Ćirić [5] developed influential quasi-contractive formulations, while nonlinear contraction theories were developed by Boyd and Wong and by Meir and Keeler [6, 7, 8]. Further directions include multivalued mappings, following Nadler's extension, Suzuki-type completeness characterizations, and F-contractions introduced by Wardowski [9, 10, 11, 12]. The aim of the present paper is to give a self-contained and application-oriented treatment of one useful symmetric Hardy–Rogers-type condition. Rather than introducing a new metric structure, the emphasis is placed on three aspects that are useful in applications: an explicit contraction ratio for Picard iteration, computable error estimates, and stability of approximate fixed points. The principal condition considered throughout the paper is

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{4}$$

with

$$a, b, c \geq 0, \qquad a + 2b + 2c < 1. \tag{5}$$

Condition (4) contains several classical contraction mechanisms as special cases and is particularly convenient because its iterative convergence rate can be written explicitly.

2. Preliminaries and Mathematical Framework

Let X be a nonempty set.

Definition 2.1 (Metric space)**.** A function $d : X \times X \to [0, \infty)$ is called a metric if, for all $x, y, z \in X$,

$$d(x, y) = 0 \Longleftrightarrow x = y, \tag{6}$$

$$d(x, y) = d(y, x), \tag{7}$$

and

$$d(x, z) \leq d(x, y) + d(y, z). \tag{8}$$

The pair (X, d) is then called a metric space.

Definition 2.2 (Cauchy sequence). A sequence $\{x_n\} \subset X$ is Cauchy if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

whenever $m, n \geq N$.

Definition 2.3 (Complete metric space). A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Completeness is essential in the contraction argument because the iteration first produces a Cauchy sequence; completeness guarantees that its limit remains inside the underlying space.

Definition 2.4 (Fixed point). A point $p \in X$ is a fixed point of $T : X \to X$ if

$$Tp = p. \tag{9}$$

The set of all fixed points of T is denoted by

$$\mathrm{Fix}(T) = \{p \in X : Tp = p\}.$$

Definition 2.5 (Picard iteration). For a chosen $x_0 \in X$, define

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots. \tag{10}$$

The sequence $\{x_n\}$ is called the Picard sequence generated by T.

Definition 2.6 (Generalized BKC contraction). A mapping $T : X \to X$ will be called a **generalized Banach–Kannan–Chatterjea contraction**, abbreviated here as a BKC contraction, if there exist constants

$$a, b, c \geq 0$$

such that

$$a + 2b + 2c < 1 \tag{11}$$

and

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{12}$$

for all $x, y \in X$.

This terminology is used only as convenient notation in the present paper; condition (12) belongs to the broader Hardy–Rogers family of generalized contractions. Hardy and Rogers' 1973 theorem treats a more general combination of such metric terms.

Lemma 2.7 (Iterative contraction factor). *Let T satisfy (12), and let*

$$x_{n+1} = Tx_n.$$

Define

$$\delta_n = d(x_n, x_{n+1}).$$

Then

$$\delta_n \leq q\delta_{n-1}, \qquad n \geq 1, \tag{13}$$

where

$$q = \frac{a + b + c}{1 - b - c}. \tag{14}$$

Moreover,

$$0 \leq q < 1. \tag{15}$$

Proof. Applying (12) to $x = x_n$ and $y = x_{n-1}$, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq ad(x_n, x_{n-1}) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + c[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]. \end{aligned} \tag{16}$$

Since

$$d(x_n, x_n) = 0$$

and, by the triangle inequality,

$$d(x_{n-1}, x_{n+1}) \leq \delta_{n-1} + \delta_n, \tag{17}$$

equation (16) gives

$$\delta_n \leq (a + b + c)\delta_{n-1} + (b + c)\delta_n. \tag{18}$$

Hence,

$$(1 - b - c)\delta_n \leq (a + b + c)\delta_{n-1}. \tag{19}$$

Therefore,

$$\delta_n \leq \frac{a + b + c}{1 - b - c}\delta_{n-1} = q\delta_{n-1}.$$

Condition (11) gives

$$a + b + c < 1 - b - c,$$

so $q < 1$. This proves the result.

Repeated application yields

$$\delta_n \leq q^n \delta_0. \tag{20}$$

□

3. Generalized Fixed-Point Theorem

Theorem 3.1 (Generalized fixed-point principle)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ satisfy the generalized contraction condition*

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{21}$$

for all $x, y \in X$, where

$$a, b, c \geq 0, \qquad a + 2b + 2c < 1. \tag{22}$$

Then:

1 *T has a fixed point $p \in X$;*
2 *the fixed point is unique;*
3 *for every starting point $x_0 \in X$, the Picard sequence*

$$x_{n+1} = Tx_n$$

converges to p.

Proof. Let $x_0 \in X$ be arbitrary and define $x_{n+1} = Tx_n$. From Lemma 2.7,

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1), \tag{23}$$

where $0 \leq q < 1$.

For $m > n$, the triangle inequality gives

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}). \tag{24}$$

Using (23),

$$d(x_n, x_m) \leq d(x_0, x_1) \sum_{k=n}^{m-1} q^k. \tag{25}$$

Thus

$$d(x_n, x_m) \leq \frac{q^n}{1-q} d(x_0, x_1). \tag{26}$$

Since $q^n \to 0$, the sequence $\{x_n\}$ is Cauchy.

Because X is complete, there exists $p \in X$ such that

$$x_n \to p. \tag{27}$$

We now prove that p is a fixed point.

Using (21) with $x = x_n$ and $y = p$,

$$\begin{aligned} d(x_{n+1}, Tp) &= d(Tx_n, Tp) \\ &\leq ad(x_n, p) + b[d(x_n, x_{n+1}) + d(p, Tp)] \\ &\quad + c[d(x_n, Tp) + d(p, x_{n+1})]. \end{aligned} \tag{28}$$

Letting $n \to \infty$, we have

$$d(x_n, p) \to 0, \qquad d(x_n, x_{n+1}) \to 0,$$

and

$$d(p, x_{n+1}) \to 0.$$

Furthermore,

$$d(x_n, Tp) \to d(p, Tp)$$

and

$$d(x_{n+1}, Tp) \to d(p, Tp).$$

Consequently,

$$d(p, Tp) \leq (b + c)d(p, Tp). \tag{29}$$

Since $b + c < 1$,

$$d(p, Tp) = 0.$$

Therefore,

$$Tp = p. \tag{30}$$

Hence p is a fixed point.

To prove uniqueness, suppose p and r are two fixed points. Then

$$Tp = p, \qquad Tr = r.$$

Applying (21),

$$d(p, r) = d(Tp, Tr) \leq ad(p, r) + b[0 + 0] + c[d(p, r) + d(r, p)].$$

Hence

$$d(p, r) \leq (a + 2c)d(p, r). \tag{31}$$

Since

$$a + 2c \le a + 2b + 2c < 1,$$

it follows that

$$d(p, r) = 0.$$

Thus $p = r$.

Therefore T has exactly one fixed point, and the Picard iteration converges to it from every initial point. □

Corollary 3.2 (Banach contraction principle). *Set $b = c = 0$. Then condition (21) reduces to*

$$d(Tx, Ty) \le ad(x, y), \qquad 0 \le a < 1. \tag{32}$$

Therefore Theorem 3.1 reduces to Banach's contraction principle. Banach's original 1922 paper is the classical source of the contraction method.

Corollary 3.3 (Kannan fixed-point theorem). *Set $a = c = 0$. Then*

$$d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)] \tag{33}$$

with $2b < 1$, or $b < \frac{1}{2}$. Thus Theorem 3.1 contains the classical Kannan contraction condition. The standard Kannan theorem uses precisely a coefficient below 1/2.

Corollary 3.4 (Chatterjea-type theorem). *Setting $a = b = 0$ gives*

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)], \qquad c < \frac{1}{2}. \tag{34}$$

This is the classical Chatterjea-type contractive form.

Example 3.5. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define

$$T(x) = 0.6x + 0.8. \tag{35}$$

Then $|T(x) - T(y)| = 0.6|x - y|$. Thus T satisfies Theorem 3.1 with

$$a = 0.6, \qquad b = c = 0.$$

The unique fixed point satisfies

$$x = 0.6x + 0.8.$$

Hence

$$0.4x = 0.8$$

and therefore

$$x^* = 2. \tag{36}$$

Starting with $x_0 = 8$,

$$x_n = 2 + 6(0.6)^n. \tag{37}$$

Thus

$$|x_n - 2| = 6(0.6)^n, \tag{38}$$

which demonstrates geometric convergence.

4. Convergence Rate, Error Bounds, and Stability

The existence of a fixed point is only one aspect of an iterative theorem. For computational applications, it is equally important to know how rapidly the iteration converges and how accurately a finite iterate approximates the exact fixed point.

Theorem 4.1 (A priori error estimate)**.** *Under the assumptions of Theorem 3.1,*

$$d(x_n, p) \le \frac{q^n}{1-q} d(x_0, x_1), \tag{39}$$

where

$$q = \frac{a+b+c}{1-b-c}.$$

Proof. For $m > n$,

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Using

$$d(x_k, x_{k+1}) \le q^k d(x_0, x_1),$$

we obtain

$$d(x_n, x_m) \le d(x_0, x_1) \sum_{k=n}^{m-1} q^k.$$

Letting $m \to \infty$ and using $x_m \to p$,

$$d(x_n, p) \le d(x_0, x_1) \sum_{k=n}^{\infty} q^k.$$

Therefore,

$$d(x_n, p) \le \frac{q^n}{1-q} d(x_0, x_1).$$

□

Corollary 4.2 (A posteriori estimate)**.** *For every n,*

$$d(x_n, p) \le \frac{1}{1-q} d(x_n, x_{n+1}). \tag{40}$$

This bound is particularly useful numerically because it depends only on consecutive iterates rather than on the unknown exact solution.

Theorem 4.3 (Ulam–Hyers-type stability)**.** *Let p denote the unique fixed point of T. Suppose $z \in X$ is an approximate fixed point satisfying*

$$d(z, Tz) \le \varepsilon. \tag{41}$$

Then

$$d(z, p) \le \frac{1-c+b}{1-a-2c} \varepsilon. \tag{42}$$

Proof. Let $D = d(z, p), \qquad r = d(z, Tz)$. Since $Tp = p$, condition (21) gives

$$d(Tz, p) = d(Tz, Tp) \le aD + b[r + 0] + c[D + d(p, Tz)].$$

Hence

$$(1-c)d(Tz, p) \le (a+c)D + br. \tag{43}$$

By the triangle inequality,

$$D \leq r + d(Tz, p). \tag{44}$$

Combining (43) and (44),

$$D \leq r + \frac{(a+c)D + br}{1-c}.$$

Multiplying by $1-c$,

$$(1-c)D \leq (1-c)r + (a+c)D + br.$$

Therefore,

$$(1-a-2c)D \leq (1-c+b)r.$$

Since $r \leq \varepsilon$,

$$D \leq \frac{1-c+b}{1-a-2c}\,\varepsilon.$$

The denominator is positive because

$$a + 2c < a + 2b + 2c < 1.$$

Thus the result follows. □

Remark 4.4 (Interpretation). Theorem 4.3 states that a small fixed-point residual

$$d(z, Tz)$$

guarantees that z lies within an explicitly controlled distance of the true fixed point.

This property is important in numerical computations where exact equality

$$Tz = z$$

is replaced by a tolerance criterion.

5. Application to Nonlinear Integral Equations

Consider the nonlinear Fredholm integral equation

$$u(t) = g(t) + \lambda \int_0^1 K(t,s)F(s, u(s))\,ds, \qquad t \in [0,1]. \tag{45}$$

Let

$$X = C([0,1], \mathbb{R})$$

equipped with the supremum metric

$$d(u,v) = \|u - v\|_\infty = \max_{0\leq t\leq 1} |u(t) - v(t)|. \tag{46}$$

Since $C([0,1])$ equipped with the supremum norm is complete, it is suitable for applying a contraction theorem.

Define

$$(Tu)(t) = g(t) + \lambda \int_0^1 K(t,s)F(s, u(s))\,ds. \tag{47}$$

Assume:

$$|F(s,u) - F(s,v)| \leq L|u - v| \tag{48}$$

for all admissible s, u, v, and suppose

$$\sup_{t\in[0,1]} \int_0^1 |K(t,s)|\, ds \le M. \tag{49}$$

Then

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\le |\lambda| \int_0^1 |K(t,s)|\, |F(s,u(s)) - F(s,v(s))|\, ds \\ &\le |\lambda| L \int_0^1 |K(t,s)|\, |u(s) - v(s)|\, ds \\ &\le |\lambda| LM \|u - v\|_\infty. \end{aligned} \tag{50}$$

Taking the supremum,

$$\|Tu - Tv\|_\infty \le |\lambda| LM \|u - v\|_\infty. \tag{51}$$

Therefore, if

$$|\lambda| LM < 1, \tag{52}$$

then T is a Banach contraction and consequently also belongs to the generalized framework of Theorem 3.1 with

$$a = |\lambda| LM, \qquad b = c = 0.$$

Hence equation (45) has a unique solution.

Theorem 5.1 (Integral-equation existence theorem)**.** *If $g \in C[0,1]$, K is such that (49) holds, F satisfies the Lipschitz condition (48), and*

$$|\lambda| LM < 1,$$

then the nonlinear integral equation (45) possesses exactly one continuous solution.

Moreover, starting from any

$$u_0 \in C[0,1],$$

the iteration

$$u_{n+1}(t) = g(t) + \lambda \int_0^1 K(t,s) F(s, u_n(s))\, ds \tag{53}$$

converges uniformly to that solution.

This illustrates why fixed-point methods are so frequently connected with integral equations: Banach's original contraction framework itself was developed in connection with equations of this broad type.

Example 5.2. Consider

$$u(t) = 1 + \frac{1}{4} \int_0^1 ts \sin(u(s))\, ds. \tag{54}$$

Here,

$$g(t) = 1, \qquad K(t,s) = ts, \qquad \lambda = \frac{1}{4}.$$

Because

$$|\sin u - \sin v| \le |u - v|,$$

we may take

$$L = 1.$$

Furthermore,

$$\int_0^1 |ts|\, ds = \frac{t}{2} \le \frac{1}{2}.$$

Thus
$$M = \frac{1}{2}.$$
Hence
$$|\lambda| LM = \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{1}{8} < 1. \tag{55}$$
Therefore equation (54) has a unique continuous solution.

6. Finite-Dimensional and Numerical Applications

Consider the vector equation
$$x = Ax + b, \tag{56}$$
where
$$A = \begin{pmatrix} 0.20 & 0.10 \\ 0.10 & 0.25 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}. \tag{57}$$
Let
$$T(x) = Ax + b.$$
Using the infinity norm,
$$\|A\|_\infty = \max\{0.20 + 0.10, 0.10 + 0.25\} = 0.35. \tag{58}$$
Therefore,
$$\|Tx - Ty\|_\infty \leq 0.35 \|x - y\|_\infty. \tag{59}$$
Since
$$0.35 < 1,$$
T is a contraction.

Consequently, the system has a unique fixed point.

Solving
$$(I - A)x = b$$
gives approximately
$$x^* = \begin{pmatrix} 1.355932 \\ 0.847458 \end{pmatrix}. \tag{60}$$
The iterative procedure
$$x_{n+1} = Ax_n + b \tag{61}$$
converges to this solution for every initial vector $x_0 \in \mathbb{R}^2$.

More generally, if $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^m$ satisfy $\|A\|_\infty < 1$, then the mapping $T(x) = Ax + b$ is a contraction on the complete metric space $(\mathbb{R}^m, \|\cdot\|_\infty)$. Hence the fixed point is unique and is given by $x^* = (I - A)^{-1} b$. Moreover, the Picard iteration $x_{n+1} = Ax_n + b$ converges geometrically, and Theorem 4.1 yields the computable estimate
$$\|x_n - x^*\|_\infty \leq \frac{\|A\|_\infty^n}{1 - \|A\|_\infty} \|x_1 - x_0\|_\infty.$$
For the matrix in (57), the contraction factor is 0.35, so the error is reduced at least geometrically according to this factor.

6.1. Numerical convergence illustration

For the scalar mapping

$$T(x) = 0.6x + 0.8,$$

with

$$x_0 = 8$$

and fixed point

$$x^* = 2,$$

we have

$$|x_n - x^*| = 6(0.6)^n.$$

The first few values are represented in Table 1. In fact, for the family $T_\lambda(x) = \lambda x + 2(1-\lambda)$ with $x_0 = 8$, every mapping has the same fixed point $x^* = 2$, and the exact error after n iterations is

$$|x_n - x^*| = 6\lambda^n.$$

Thus the contraction parameter directly controls the observed convergence speed.

Table 1: Comparison of convergence rates for generalized contraction mappings in a complete metric space.

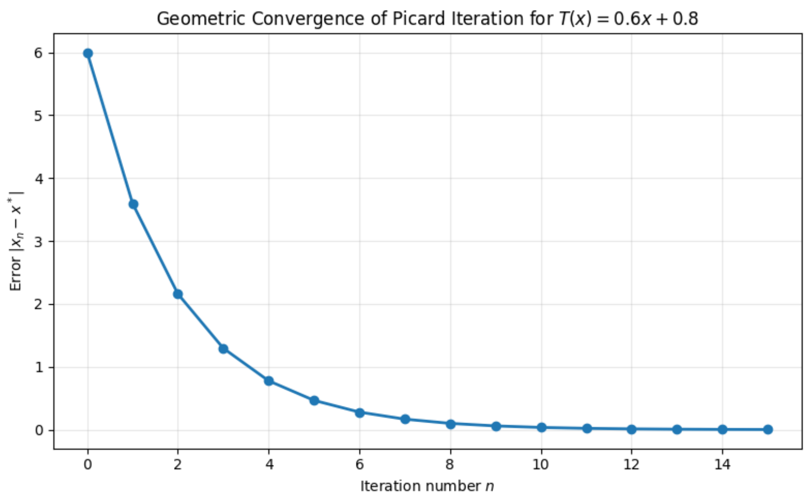
Mapping	Contraction parameter (λ)	Initial value (x_0)	x_5	Error after 5 iterations	Convergence
$T_1(x) = 0.2x + 1.6$	0.20	8.00	2.001920	0.001920	Very fast
$T_2(x) = 0.4x + 1.2$	0.40	8.00	2.061440	0.061440	Fast
$T_3(x) = 0.6x + 0.8$	0.60	8.00	2.466560	0.466560	Moderate
$T_4(x) = 0.7x + 0.6$	0.70	8.00	3.008420	1.008420	Slow
$T_5(x) = 0.8x + 0.4$	0.80	8.00	3.966080	1.966080	Very slow

The table shows the expected monotone relationship between the contraction parameter and the finite-iteration error: smaller values of λ give substantially faster convergence, while values closer to 1 produce slower geometric decay. This is exactly the behavior predicted by the contraction theory.

The generated graph illustrates the geometric reduction of

$$|x_n - x^*|$$

as predicted theoretically by the contraction factor $q = 0.6$.



Geometric convergence of Picard iteration for $T(x) = 0.6x + 0.8$.

The numerical behavior is consistent with estimate (39): increasing the iteration index reduces the error geometrically. In particular, the numerical table and graph confirm that the theoretical contraction factor is not merely qualitative: it provides a direct quantitative description of the speed of convergence and can be used to estimate the number of iterations required to achieve a prescribed tolerance.

7. Conclusion

This study presented a unified fixed-point framework for a class of generalized contractions in complete metric spaces. The principal condition,

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)],$$

was investigated under the parameter restriction

$$a + 2b + 2c < 1.$$

The central iterative quantity was shown to satisfy

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1),$$

where

$$q = \frac{a + b + c}{1 - b - c} < 1.$$

This estimate establishes that the Picard sequence is Cauchy. Completeness of the metric space consequently yields convergence to a point p, and the generalized contractive condition proves that

$$Tp = p.$$

A separate argument establishes uniqueness.

The theorem simultaneously recovers the classical Banach, Kannan, and Chatterjea contraction principles through appropriate choices of a, b, and c. Such unification is closely related to the broader Hardy–Rogers tradition of combining several metric terms in a single contractive inequality.

In addition to existence and uniqueness, two quantitative error bounds were established. The a priori estimate

$$d(x_n, p) \leq \frac{q^n}{1 - q} d(x_0, x_1)$$

predicts convergence before computation, while the a posteriori estimate

$$d(x_n, p) \leq \frac{1}{1 - q} d(x_n, x_{n+1})$$

allows a stopping criterion based on computed iterates.

An Ulam–Hyers-type estimate further showed that an approximate fixed point satisfying

$$d(z, Tz) \leq \varepsilon$$

remains quantitatively close to the exact solution.

Applications to nonlinear Fredholm integral equations and finite-dimensional iterative systems demonstrated how the abstract metric theorem can be translated directly into solvability conditions. The numerical example confirmed the predicted geometric convergence of Picard iteration.

Future work may investigate analogous conditions for multivalued mappings, ordered metric spaces, b-metric spaces, generalized metric structures, and nonlinear F-contractions. Nadler's multivalued contraction theory, Suzuki's completeness characterizations, and Wardowski's F-contraction framework demonstrate established directions in which classical metric fixed-point theory has been extended.

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