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Computation for Lucky Number of Some Families of Planar Graphs

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Abstract

Let G be a finite simple graph and let $\lambda(G) \to 1, 2, \ldots, k$ be a vertex labeling. For each vertex $v \in V(G)$, define the induced neighborhood sum by $S_\lambda(v) = \sum_{u \in N(v)} \lambda(u)$. The labeling λ is called a lucky labeling if $S_\lambda(u) \neq S_\lambda(v)$ for every edge $uv \in E(G)$, and the lucky number $\eta(G)$ is the minimum value of k for which such a labeling exists. In this paper, we determine the lucky numbers of several structured families of planar graphs, including pentagonal, alternate pentagonal, and double pentagonal snake graphs; pentagonal book graphs; hairy cycle graphs; Jahangir graphs; and quadrilateral snake graphs with pendant edges together with their alternate forms. For each family, explicit vertex labelings are constructed and the corresponding neighborhood sums are evaluated to verify the lucky condition. The obtained lucky numbers are uniformly small, lying between 2 and 4, and within the considered families they depend at most on simple parity conditions rather than on the order of the graph. These constructions extend the catalogue of planar graphs with known exact lucky numbers and provide labeling patterns that may be adapted to related graph families.

Keywords: Lucky Number, Graph Theory, Labeling, Planar graphs, Pentagonal Snake Graph
2010 Mathematics Subject Classification: Primary 05C78; Secondary 05C15, 05C10.

1. Introduction

Graph labeling assigns integers or other symbols to the vertices, edges, or both elements of a graph subject to prescribed conditions. Among the many labeling schemes, lucky labeling is distinguished by the fact that the labels themselves need not form a proper coloring; instead, they induce a proper coloring through sums taken over open neighborhoods. More precisely, for a labeling $\lambda : V(G) \to 1, 2, \ldots, k$, the induced value at a vertex v is

$$S_\lambda(v) = \sum_{u \in N(v)} \lambda(u),$$

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and λ is lucky when $S_\lambda(u) \neq S_\lambda(v)$ for every edge $uv \in E(G)$. The least admissible value of k is the lucky number $\eta(G)$. In the contemporary literature, the same concept is also called *additive coloring*, and $\eta(G)$ is often referred to as the additive chromatic number [4, 23, 8].

Lucky labeling was introduced by Czerwi'nski, Grytczuk, and Zelazny [4], who also formulated the additive coloring conjecture

$$\eta(G) \leq \chi(G).$$

The subject quickly developed in both structural and computational directions. Ahadi, Dehghan, Kazemi, and Mollaahmadi [1] proved that deciding whether a planar 3-colorable graph has lucky number 2 is NP-complete, demonstrating that exact computation remains difficult even under strong restrictions. Akbari et al. [2] studied the list version through the lucky choice number, while Sever'in [23] obtained exact values for several graph families and verified the additive coloring conjecture for additional classes. More recently, Gossett [8] established an Alon–Tarsi-type result for additive list colorings, connecting lucky labeling with orientation methods and algebraic techniques in graph coloring.

Alongside these general developments, considerable attention has been given to explicit constructions for special graph families. Lucky and proper lucky labelings of quadrilateral snake graphs were investigated by Sateesh Kumar and Meenakshi [21], and Indira, Selvam, and Thirusangu [13] treated extended duplicate graphs derived from quadrilateral snakes. Sateesh Kumar and Meenakshi also determined lucky labelings for jellyfish, cocktail-party, and crown graphs [22]. Ashwini, Pethanachi Selvam, and Gnanajothi [3] obtained further exact results for complete, complete bipartite, and complete tripartite graphs, whereas Kujur [16] studied proper d-lucky numbers of rooted product graphs. Recent work has also continued the computation of exact lucky numbers for comb and triangular-snake-related families [12]. These studies show that constructive arguments remain an effective method for identifying exact lucky numbers in highly structured graphs.

Motivated by this line of research, the present paper studies a collection of planar graph families for which compact lucky labelings can be described explicitly. We consider pentagonal snake graphs, alternate pentagonal snake graphs, double pentagonal snake graphs, books with pentagonal pages, hairy cycle graphs $C_m \odot 3K_1$, Jahangir graphs $J_{n,m}$, quadrilateral snake graphs with pendant edges, and alternate quadrilateral snake graphs with pendant edges. For each family, a labeling is specified and the induced neighborhood sums are computed in order to verify that adjacent vertices receive distinct sums. The resulting formulas show that the lucky numbers of these families are bounded by small constants and, in the parity-dependent cases, are independent of the graph order apart from whether the relevant parameter is even or odd.

The remainder of the paper is organized as follows. Section 2 recalls the necessary terminology and notation. Section 3 presents the labeling constructions and the corresponding lucky-number computations. The final section summarizes the results and indicates possible extensions to other planar graph families and network structures.

2. Preliminaries

Definition 2.1. Let $G(V, E)$ be a graph with vertex set V and edge set E, then the process of assigning integers to a vertex set, an edge set or both is known as graph labeling.

Definition 2.2. [21] Let $\lambda : V(G) \to \{1, 2, ..., z\}$ be a labeling of vertices V of a graph G. A labeling λ is said to be lucky if $S(u) \neq S(v)$ for an adjacent vertex u and v, where $S(v)$ and $S(u)$ denote the sum of labels over the neighbours of the vertex v and u in G. If v is an isolated vertex of G then $S(v) = 0$.

Definition 2.3. If a graph admits lucky labeling then this graph is known as lucky labeled graph.

Definition 2.4. The Lucky number of a graph is the least natural number which is used for lucky labeled Graph G is denoted by $\eta(G)$.

Definition 2.5. Chromatic Number is the minimum or smallest number required to colour the vertices of a graph such that no two adjacent vertices share same colour. Chromatic number is denoted by $\chi(G)$.

Definition 2.6. An edge of a graph is pendent if and only if one of it's vertices has degree one.

Theorem 2.7. *For every non-directed and simple graph G, $\chi(G) \leq \delta(G) + 1$, where $\delta(G)$ is the highest degree in G.*

Theorem 2.8. *For every planar graph G, $\eta(G) \leq \chi(G)$.*

3. Main Results

Theorem 3.1. *The pentagonal snake graph PSn, $n > 1$ is lucky labeled graph with*

$$\eta(PS_n) = \begin{cases} 2 & \text{when } n \text{ is even} \\ 3 & \text{when } n \text{ is odd} \end{cases}$$

Proof. [6] Pentagonal snake graph $PS_n, (n > 1)$ can be obtained by vertex set, $V(PS_n)= \{u_i; 1 \leq i \leq n\} \cup \{u_i, v_i; 1 \leq i \leq n-1\}$ and edge set,$E(PS_n) = \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_iv_i; 1 \leq i \leq n-1\} \cup \{v_ix_i; 1 \leq i \leq n-1\} \cup \{v_ix_i; 1 \leq i \leq n-1\} \cup \{v_ix_i; 1 \leq i \leq n-1\}$. Because, the highest degree of pentagonal snake graph $PS_n, n > 1$ is 4.So by theorem 2.7.$\chi(PS_n) \leq 5$. But for $n = 2$ the highest degree is 2. so again by theorem 2.7. $\chi(PS_2) \leq 3$. If we assign less than three colour then adjacent vertices can't have different colours. By assigning exactly three colours adjacent vertices have different colours, so PS_n is 3 colourable. Hence $\chi(PS_n) = 3$.By theorem 2.8 we found upper bound $\eta(PS_n) \leq 3$. Now, we define vertex labeling, when n is even

$$\lambda : V(PS_n) \to \{1, 2, 3\}$$

such that

$$\lambda(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

$$\lambda(u_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 1 \leq i \leq n, \text{even} \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 3, & 2 \leq i \leq n-2, \text{even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 2 \leq i \leq n-2, \text{even} \\ 3, & 1 \leq i \leq n-1, \text{odd} \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(x_i) = \begin{cases} 4, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(u_i) = \begin{cases} 3, & i = 1 \\ 4, & i = n \\ 6, & 3 \leq i \leq n-1, \text{odd} \\ 8, & 2 \leq i \leq n-2, \text{even} \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & 1 \leq i \leq n-1, \text{odd} \\ 3, & 2 \leq i \leq n-2, \text{even} \end{cases}$$

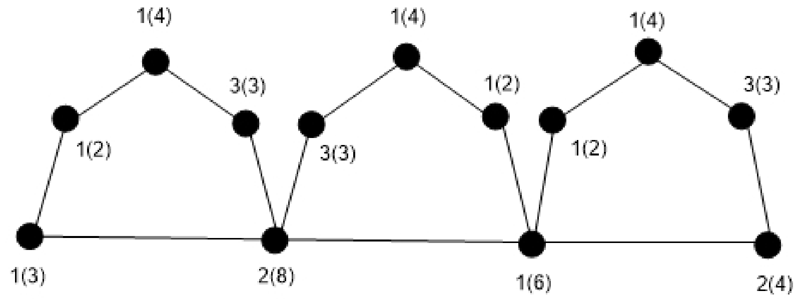

Figure 1: Pentagonal Snake Graph PS_4

$$\mu(w_i) = \begin{cases} 2, & 2 \leq i \leq n-2, \text{even} \\ 3, & 1 \leq i \leq n-1, \text{odd} \end{cases}$$

We see that the sum of adjacent vertices are not identical. So, $PS_n, n > 1$ for $n = even$ is lucky labeled graph with $\eta(PS_n) = 3$.

Now, when n is odd, lucky labeling define as

$$\lambda : V(PS_n) \rightarrow \{1, 2\}$$

such that

$$\lambda(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

$$\lambda(u_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 2 \leq i \leq n-1, \text{even} \\ 2, & 1 \leq i \leq n-2, \text{odd} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 1 \leq i \leq n-2, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

Now, we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(x_i) = \begin{cases} 3, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(u_i) = \begin{cases} 3, & i = 1, n \\ 4, & 2 \leq i \leq n-1, \text{even} \\ 6, & 3 \leq i \leq n-2, \text{odd} \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(x_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

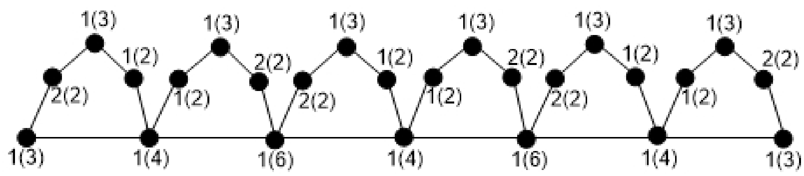

Figure 2: Pentagonal Snake Graph PS_7

We see that the sum of adjacent vertices are not identical. So, $PS_n, n > 1$ for $n = odd$ is lucky labeled graph with $\eta(PS_n) = 2$.
Hence, pentagonal snake graph PSn, $n > 1$ is lucky labeled graph with

$$\eta(PS_n) = \begin{cases} 2 & \text{when n is even} \\ 3 & \text{when n is odd} \end{cases}$$

□

Theorem 3.2. *Alternate pentagonal snake graph $APS_n, n \leq 2$ is lucky labeled graph with $\eta(APS_n) = 3$.*

Proof. [6] Alternate pentagonal snake graph $APS_n, n \leq 2$ can be constructed by the vertex set $V(APS_n) = \{u_i, u_i^{'}; 1 \leq i \leq n\} \cup \{v_i, w_i, x_i; 1 \leq i \leq n\}$ and edge set $E(APS_n) = \{u_i u_i^{'}; 1 \leq i \leq n\} \cup \{u_i^{'} u_{i+1}; 1 \leq i \leq n\} \cup \{u_i v_i; 1 \leq i \leq n\} \cup \{v_i, x_i; 1 \leq i \leq n\} \cup \{x_i w_i; 1 \leq i \leq n\} \cup \{w_i u_i^{'}; 1 \leq i \leq n\}$. As the highest degree of vertex of the Alternate pentagonal snake graph $APS_n, n \leq 2$ is 3. So, by theorem 2.7. $\chi(APS_n) \leq 4$. If we assign less than three colour then adjacent vertices can't have different colours. By assigning exactly three colours adjacent vertices have different colours, so APS_n is 3 colourable. Hence $\chi(PS_n) = 3$. By theorem 2.8 we found upper bound $\eta(APS_n) \leq 3$. Now, we define vertex labeling,

$$\lambda : V(PS_n) \rightarrow \{1, 2, 3\}$$

such that

$$\lambda(x_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(u_i) = \begin{cases} 2, & 1 \leq i \leq n \end{cases}$$

$$\lambda(u_i^{'}) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(w_i) = \begin{cases} 3, & 1 \leq i \leq n \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(x_i) = \begin{cases} 4, & 1 \leq i \leq n \end{cases}$$

$$\mu(u_i) = \begin{cases} 2, & i = 1 \\ 3, & 2 \leq i \leq n \end{cases}$$

$$\mu(u_i^{'}) = \begin{cases} 5, & i = n \\ 7, & 2 \le i \le n-1 \end{cases}$$

$$\mu(v_i) = \begin{cases} 3, & 1 \le i \le n \end{cases}$$

$$\mu(w_i) = \begin{cases} 2, & 1 \le i \le n \end{cases}$$

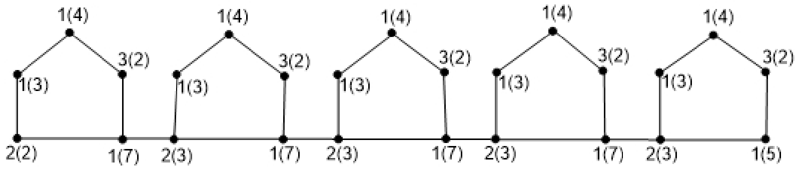

Figure 3: Alternate Pentagonal Snake Graph PS_7

We see that the sum of adjacent vertices are not identical.
Hence, alternate pentagonal snake graph $APSn, n > 1$ is lucky labeled graph with $\eta(APS_n) = 3$. □

Theorem 3.3. *Double pentagonal snake graph $DPS_n, n \ge 2$ is lucky labeled graph with $\eta(DPS_n) = 2$.*

Proof. [6] Double pentagonal snake graph $DPS_n, n \ge 2$ can be constructed by the vertex set $V(DPS_n) = \{u_i; 1 \le i \le n\} \cup \{v_i, w_i, x_i; 1 \le i \le n-1\} \cup \{v_i^{'}, w_i^{'}, x_i^{'}; 1 \le i \le n-1\}$ and edge set $E(DPS_n) = \{u_i u_{i+1}; 1 \le i \le n-1\} \cup \{u_i v_i; 1 \le i \le n-1\} \cup \{v_i x_i; 1 \le i \le n-1\} \cup \{x_i, w_i; 1 \le i \le n-1\} \cup \{w_i u_{i+1}; 1 \le i \le n-1\} \cup \{u_i v_i^{'}; 1 \le i \le n\} \cup \{v_i^{'} x_i^{'}; 1 \le i \le n-1\} \cup \{x_i^{'} w_i^{'}; 1 \le i \le n-1\} \cup \{w_i^{'} u_{i+1}; 1 \le i \le n\}$. As the highest degree of vertex of the double pentagonal snake graph $DPS_n, n \ge 2$ is 6. So, by theorem 2.7. $\chi(DPS_n) \le 7$. If we assign less than three colour then adjacent vertices can't have different colours. By assigning exactly three colours adjacent vertices have different colours, so DPS_n is 3 colourable. Hence $\chi(DPS_n) = 3$. By theorem 2.8 we found upper bound $\eta(DPS_n) \le 3$. Now, we define vertex labeling,

$$\lambda : V(DPS_n) \to \{1, 2\}$$

such that

When n is even, $(n \ge 2)$

$$\lambda(u_i) = \begin{cases} 1, & 1 \le i \le n \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \le i \le n-1, \text{odd} \\ 2, & 2 \le i \le n-2, \text{even} \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 1 \le i \le n-1, \text{odd} \\ 2, & 2 \le i \le n-2, \text{even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 2 \le i \le n-2, \text{even} \\ 2, & 1 \le i \le n-1, \text{odd} \end{cases}$$

$$\lambda(w_i^{'}) = \begin{cases} 1, & 2 \le i \le n-2, \text{even} \\ 2, & 1 \le i \le n-1, \text{odd} \end{cases}$$

$$\lambda(x_i) = \begin{cases} 1, & 1 \le i \le n-1 \end{cases}$$

$$\lambda(x_i^{'}) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(u_i) = \begin{cases} 3, & i = 1 \\ 5, & i = n \\ 6, & 3 \leq i \leq n-1, \text{odd} \\ 10, & 2 \leq i \leq n-2, \text{even} \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(v_i^{'}) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(w_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(w_i^{'}) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(x_i) = \begin{cases} 3, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(w_i^{'}) = \begin{cases} 3, & 1 \leq i \leq n-1 \end{cases}$$

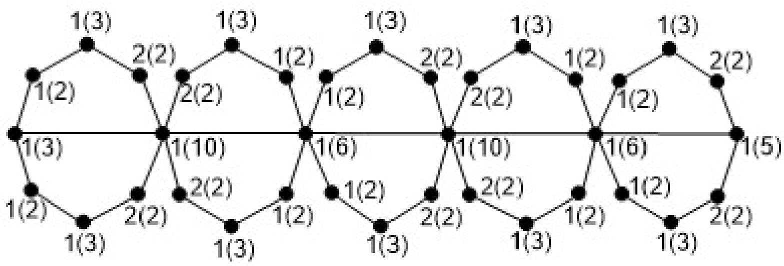

Figure 4: Double Pentagonal Snake Graph DPS_6

When n is odd, $(n \geq 3)$

$$\lambda(u_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq n-2, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 1 \leq i \leq n-2, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 2 \leq i \leq n-1, \text{even} \\ 2, & 1 \leq i \leq n-2, \text{odd} \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 2 \leq i \leq n-1, \text{even} \\ 2, & 1 \leq i \leq n-2, \text{odd} \end{cases}$$

$$\lambda(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(u_i) = \begin{cases} 3, & i = 1, n \\ 6, & 3 \leq i \leq n-2, \text{odd} \\ 10, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(v_i^{'}) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(w_i) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(w_i^{'}) = \begin{cases} 2, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(x_i) = \begin{cases} 3, & 1 \leq i \leq n-1 \end{cases}$$

$$\mu(x_i^{'}) = \begin{cases} 3, & 1 \leq i \leq n-1 \end{cases}$$

We see that the sum of adjacent vertices are not identical.

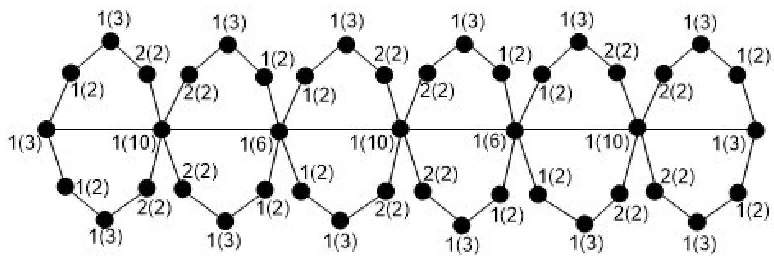

Figure 5: Double Pentagonal Snake Graph DPS_7

Hence, double pentagonal snake graph $DPSn, n > 1$ is lucky labeled graph with $\eta(DPS_n) = 2$. □

Theorem 3.4. *A book with n-pentagonal pages B_5^t is lucky labeled with $\eta(B_5^t) = 3$.*

Proof. [17] A book with n-pentagonal pages B_5^t can be constructed by the vertex set $V(B_5^t) = \{u, v, u_i, v_i, w_i; 1 \leq i \leq n\}$ and edge set $E(B_5^t) = \{uv\} \cup \{uu_i; 1 \leq i \leq n\} \cup \{u_iw_i; 1 \leq i \leq n\} \cup \{vv_i; 1 \leq i \leq n\} \cup \{v_iw_i; 1 \leq i \leq n\}$. As the highest degree of vertex of the book with n-pentagonal pages B_5^t is $n+1$. So, by theorem 2.7. $\chi(B_5^t) \leq n+2$. If we assign less than three colour then adjacent vertices can't have different colours.

By assigning exactly three colours adjacent vertices have different colours, so B_5^t is 3 colourable. Hence $\chi(B_5^t) = 3$. By theorem 2.8 we found upper bound $\eta(B_5^t) \leq 3$. Now, we define vertex labeling,

$$\lambda : V(B_5^t) \to \{1,2,3\}$$

such that

If n is odd, $(n \geq 1)$

$$\lambda(u) = \begin{cases} 1 \end{cases}$$

$$\lambda(v) = \begin{cases} 2 \end{cases}$$

$$\lambda(u_i) = \begin{cases} 1, & 1 \leq i \leq n, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq n, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 2 \leq i \leq n-1, \text{even} \\ 3, & 1 \leq i \leq n, \text{odd} \end{cases}$$

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(u) = \begin{cases} 3 + 3(n-1)/2 \end{cases}$$

$$\mu(v) = \begin{cases} 3(n-1)/2 + 2 \end{cases}$$

$$\mu(u_i) = \begin{cases} 4, & 1 \leq i \leq n, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{odd} \end{cases}$$

$$\mu(v_i) = \begin{cases} 5, & 1 \leq i \leq n, \text{odd} \\ 3, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\mu(w_i) = \begin{cases} 2, & 1 \leq i \leq n, \text{odd} \\ 4, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

If n is even, $(n > 1)$

$$\lambda(u) = \begin{cases} 1 \end{cases}$$

$$\lambda(v) = \begin{cases} 2 \end{cases}$$

$$\lambda(u_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\lambda(w_i) = \begin{cases} 1, & 2 \leq i \leq n, \text{even} \\ 3, & 1 \leq i \leq n-1, \text{odd} \end{cases}$$

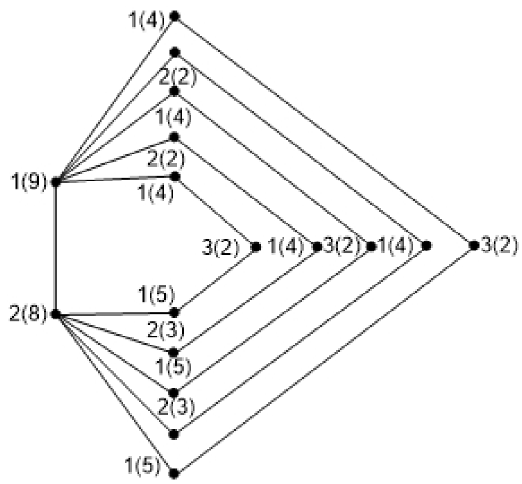

Figure 6: t Pentagonal Pages Book Graph B_7^t

Now we evaluate sum of neighbour vertices denoted by μ as follow,

$$\mu(u) = \begin{cases} 3 + 1/2 + 3(n-1)/2 \end{cases}$$

$$\mu(v) = \begin{cases} 3(n)/2 + 1 \end{cases}$$

$$\mu(u_i) = \begin{cases} 4, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\mu(v_i) = \begin{cases} 5, & 1 \leq i \leq n-1, \text{odd} \\ 3, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\mu(w_i) = \begin{cases} 2, & 1 \leq i \leq n-1, \text{odd} \\ 4, & 2 \leq i \leq n, \text{even} \end{cases}$$

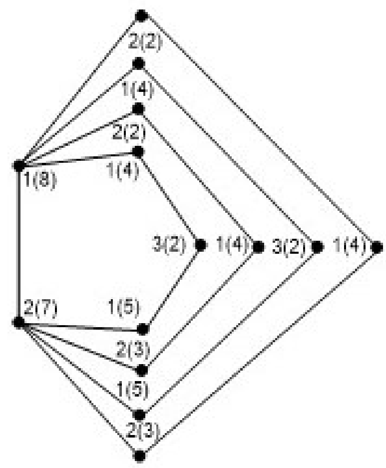

Figure 7: t Pentagonal Pages Book Graph B_4^t

We see that the sum of adjacent vertices are not identical.
Hence,A book with n-pentagonal pages B_5^t is lucky labeled with $\eta(B_5^t) = 3$. □

Theorem 3.5. *Hairy cycle graph $C_m \odot 3K_1$, $m \geq 3$ is lucky labeled graph with*

$$\eta(C_m \odot 3K_1) = \begin{cases} 3 & \text{when } m \text{ is odd} \\ 4 & \text{when } m \text{ is even} \end{cases}$$

Proof. [19] Hairy cycle graph $C_m \odot 3K_1, m \geq 3$ can be obtained by vertex set, $V(C_m \odot 3K_1) = \{v_i, v_i^{'}, v_i^{''}, v_i^{'''}; 1 \leq i \leq m\}$ and edge set $E(C_m \odot 3K_1) = \{v_i v_{i+1}; 1 \leq i \leq m-1\} \cup \{v_i v_i^{'}; 1 \leq i \leq m\} \cup \{v_i v_i^{''}; 1 \leq i \leq m\} \cup \{v_i v_i^{'''}; 1 \leq i \leq m\} \cup \{v_1 v_m\}$. As, the highest degree of vertex of the comb graph $C_m \odot 3K_1$ is 5. So, by theorem 2.7. $\chi(C_m \odot 3K_1) \leq 6$. For $m \geq 3$ odd, if we assign less than three colours then adjacent vertices can't have different colours. By assigning exactly three colours the adjacent vertices have different colours. So, the comb graph $C_m \odot 3K_1$ $m \geq 3, (odd)$ is 3 colourable. Hence, $\chi(C_m \odot 3K_1) = 3$. By theorem 2.8. we found upper bound $\eta(C_m \odot 3K_1) \leq 3$. Now we define vertex labeling for $m \geq 3$ odd

$$\lambda : V(C_m \odot 3K_1) \rightarrow \{1, 2, 3\}$$

such that

$$\lambda(v_i) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

$$\lambda(v_i^{''}) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

$$\lambda(v_i^{'''}) = \begin{cases} 1, & i = 1 \\ 2, & 2 \leq i \leq m-1, \text{even} \\ 3, & 3 \leq i \leq m, \text{odd} \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(v_i) = \begin{cases} 5, & i = 1 \\ 6, & 2 \leq i \leq m-1, \text{even} \\ 7, & 3 \leq i \leq m, \text{odd} \end{cases}$$

$$\mu(v_i^{'}) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

$$\mu(v_i^{''}) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

$$\mu(v_i^{'''}) = \begin{cases} 1, & 1 \leq i \leq m \end{cases}$$

Now, for $n \geq 4$ even, if we assign less than two colours adjacent vertices can't have different colours. By assigning exactly two colours adjacent vertices have different colours. So, comb graph $C_m \odot 3K_1$ is 2 colourable. Hence, $\chi(C_m \odot 3K_1) = 2$. By theorem 2.8 we found upper bound $\eta(C_m \odot 3K_1) \leq 2$. Now we define vertex labeling for $n \geq 4$ even

$$\lambda : V(C_m \odot 3K_1) \rightarrow \{1, 2\}$$

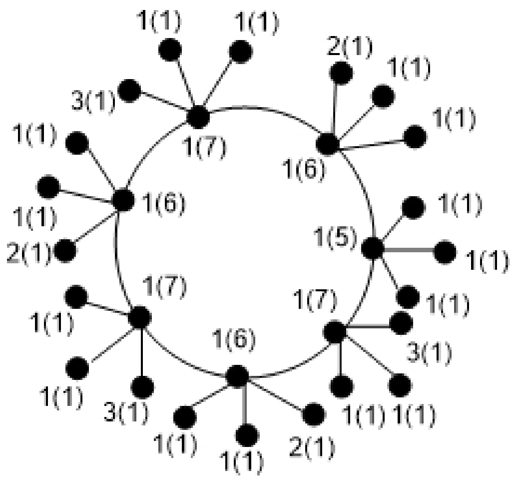

Figure 8: Lucky Labeling of Hairy Cycle Graph $C_7 \odot 3K_1$

such that

$$\lambda(v_i) = \begin{cases} 1, & 1 \le i \le m, \text{odd} \\ 2, & 2 \le i \le m-1, \text{even} \end{cases}$$

$$\lambda(v_i^{'}) = \begin{cases} 1, & 1 \le i \le m \end{cases}$$

$$\lambda(v_i^{''}) = \begin{cases} 1, & 1 \le i \le m \end{cases}$$

$$\lambda(v_i^{'''}) = \begin{cases} 1, & 1 \le i \le m \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(v_i) = \begin{cases} 5, & 2 \le i \le m, \text{even} \\ 7, & 1 \le i \le m-1, \text{odd} \end{cases}$$

$$\mu(v_i^{'}) = \begin{cases} 1, & 1 \le i \le m-1, \text{odd} \\ 2, & 2 \le i \le m, \text{even} \end{cases}$$

$$\mu(v_i^{''}) = \begin{cases} 1, & 1 \le i \le m-1, \text{odd} \\ 2, & 2 \le i \le m, \text{even} \end{cases}$$

$$\mu(v_i^{'''}) = \begin{cases} 1, & 1 \le i \le m-1, \text{odd} \\ 2, & 2 \le i \le m, \text{even} \end{cases}$$

We see that the sum of adjacent vertices are not identical.
Hence, hairy cycle graph $C_m \odot 3K_1$, $m \ge 3$ is lucky labeled graph with

$$\eta(C_m \odot 3K_1) = \begin{cases} 3 & \text{when m is odd} \\ 4 & \text{when m is even} \end{cases}$$

□

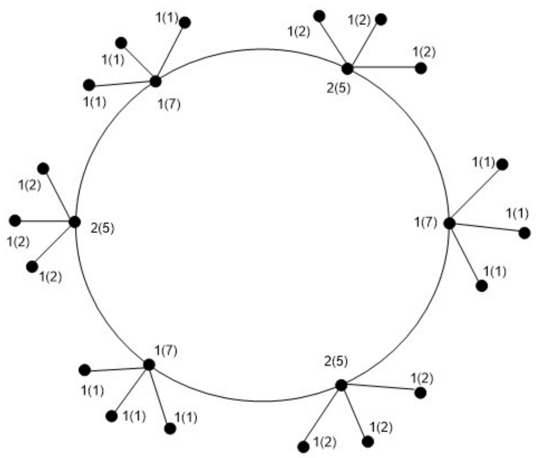

Figure 9: Lucky Labeling of Hairy Cycle Graph $C_6 \odot 3K_1$

Theorem 3.6. *Jahangir graph $J_{n,m}(n \geq 2; m \geq 3)$ is lucky labeled graph with*

$$\eta(J_{n,m}) = \begin{cases} 2 & \textit{when } n \textit{ is even}, n \geq 2 \\ 3 & \textit{when } n \textit{ is odd}, n \geq 3 \end{cases}$$

Proof. [10] Jahangir graph $J_{n,m}(n \geq 2; m \geq 3)$ can be obtained by vertex set, $V(J_{n,m}) = \{v_0\} \cup \{v_i; 1 \leq i \leq mn\}$ and edge set $E(J_{n,m}) = \{v_i v_{i+1}; 1 \leq i \leq mn-1\} \cup \{v_{mn} v_1\} \cup \{v_0 v_{1+n(i-1)}; 1 \leq i \leq m\}$. As, the highest degree of vertex of the Jahangir graph $J_{n,m}$ is m. So, by theorem 2.7. $\chi(J_{n,m}) \leq m+1$. For $n \geq 2$ even, if we assign less than two colours then adjacent vertices can't have different colours. By assigning exactly two colours adjacent vertices have different colours. So, $J_{n,m}$, $n \geq 2$ even is 2 colourable. Hence, $\chi(J_{n,m}) = 2$. By theorem 2.8 we found upper bound $\eta(J_{n,m}) \leq 2$. Now we define vertex labeling, when $n \geq 2$ even

$$\lambda : V(J_{n,m}) \to \{1, 2\}$$

such that

$$\lambda(v_0) = \begin{cases} 1 \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & 2 \leq i \leq mn, \text{even} \\ 2, & 1 \leq i \leq mn-1, \text{odd} \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(v_0) = \begin{cases} 2m \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & i = nu - (2l-1); 1 \leq l \leq (n-2)/2; 1 \leq u \leq m \\ 3, & i = nu - (n-1); 1 \leq u \leq m \\ 4, & 2 \leq i \leq mn, \text{even} \end{cases}$$

For $n \geq 3$ odd, if we assign less than three colours then adjacent vertices can't have different colours. By assigning exactly three colours adjacent vertices have different colours. So, $J_{n,m}$, $n \geq 3$ is 3 colourable.

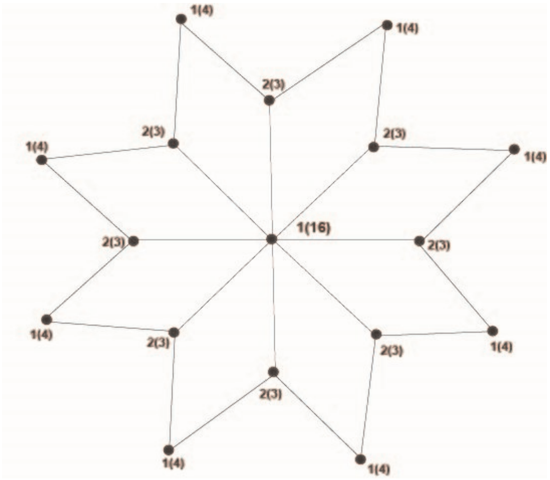

Figure 10: Jahangir Graph $J_{2,8}$

Hence, $\chi(J_{n,m}) = 3$. By theorem 2.8 we found upper bound $\eta(J_{n,m}) \leq 3$. Now we define vertex labeling for $n \geq 3$ odd (except n=3),

$$\lambda : V(J_{n,m}) \to \{1, 2, 3\}$$

$$\lambda(v_0) = \begin{cases} 1 \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & i = nu - (2l-1); 1 \leq l \leq (n-1)/2; 1 \leq u \leq m \\ 1, & i = nu; 1 \leq u \leq m \\ 2, & i = nu - 2l; 1 \leq l \leq (n-3)/2; 1 \leq u \leq m \\ 3, & i = nu - (n-1); 1 \leq u \leq m \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(v_0) = \begin{cases} 3m \end{cases}$$

$$\mu(v_i) = \begin{cases} 2, & i = nu - 2l; 1 \leq l \leq (n-3)/2; 1 \leq u \leq m \\ 3, & i = nu - (n-1); 1 \leq u \leq m \\ 3, & i = nu - 1; 1 \leq u \leq m \\ 4, & i = nu; 1 \leq u \leq m \\ 4, & i = nu - (2l+1); 1 \leq l \leq (n-5)/2; 1 \leq u \leq m \\ 5, & i = nu - (n-2); 1 \leq u \leq m \end{cases}$$

For n=3,

$$\lambda(v_0) = \begin{cases} 3 \end{cases}$$

$$\lambda(v_i) = \begin{cases} 1, & i = 3u - 1; 1 \leq u \leq m \\ 2, & i = 3u; 1 \leq u \leq m \\ 3, & i = 3u - 2; 1 \leq u \leq m \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(v_0) = \begin{cases} 3m \end{cases}$$

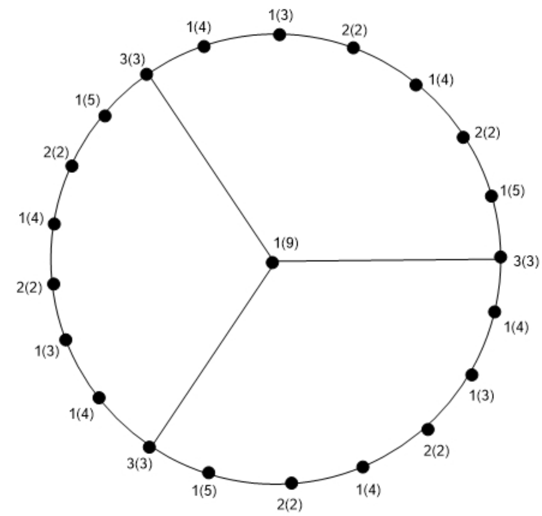

Figure 11: Jahangir Graph $J_{7,3}$

$$\mu(v_i) = \begin{cases} 4, & i = 3u; 1 \leq u \leq m \\ 5, & i = 3u-1; 1 \leq u \leq m \\ 6, & i = 3u-2; 1 \leq u \leq m \end{cases}$$

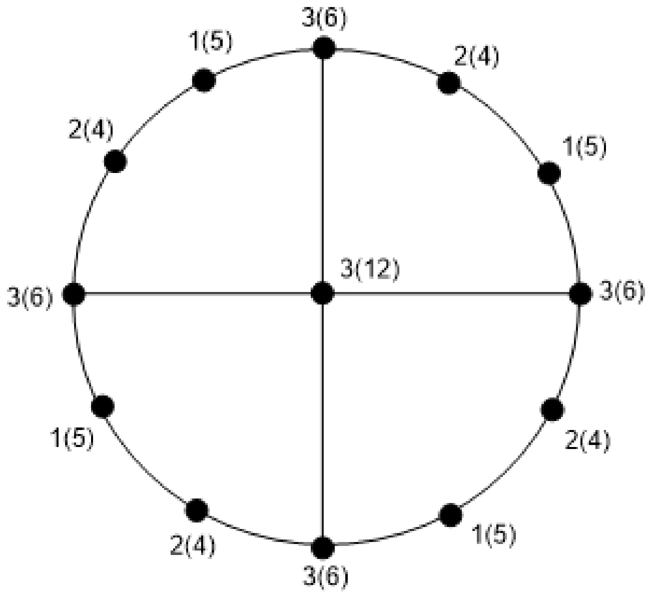

Figure 12: Jahangir Graph $J_{3,4}$

We see that the sum of adjacent vertices are not identical.
Hence, Jahangir graph $J_{n,m}(n \geq 2; m \geq 3)$ is lucky labeled graph with

$$\eta(J_{n,m}) = \begin{cases} 2 & \text{when n is even}, n \geq 2 \\ 3 & \text{when n is odd}, n \geq 3 \end{cases}$$

□

Theorem 3.7. *Quadrilateral snake graph with pendent edges $PQS_n, n \geq 1$ is lucky labeled graph with $\eta(PQS_n) = 2$.*

Proof. [11] Quadrilateral snake graph with pendent edges PQS_n can be constructed by vertex set $V(PQS_n) = \{a_i, l_i; 1 \leq i \leq 2n-2\} \cup \{b_i, k_i; 1 \leq i \leq n\}$ and edge set $E(PQS_n) = \{b_i b_{i+1}; 1 \leq i \leq n-1\} \cup \{a_{2i-1} b_i; 1 \leq i \leq n-1\} \cup \{a_{2i} b_{i+1}; 1 \leq i \leq n-1\} \cup \{a_{2i-1} a_{2i}; 1 \leq i \leq n-1\} \cup \{a_i l_i; 1 \leq i \leq 2n-2\} \cup \{b_i k_i; 1 \leq i \leq n\}$. As, the highest vertex degree of quadrilateral snake graph with pendent edges PQS_n is 5. So, by theorem 2.7 $\chi(PQS_n) \leq 6$. If we assign less than two colours then adjacent vertices can't have different colours. By assigning exactly two colours adjacent vertices have different colours. Hence, $\chi(PAS_n) = 2$. So, by theorem 2.8 we found upper bound $\eta(PQS_n) \leq 2$. Now we define vertex labeling

$$\lambda : V(PQS_n) \to \{1, 2\}$$

such that

$$\lambda(l_i) = \begin{cases} 1, & 1 \leq i \leq 2n-2 \end{cases}$$

$$\lambda(a_i) = \begin{cases} 1, & 1 \leq i \leq 2n-2, \text{odd} \\ 2, & 2 \leq i \leq 2n-2, \text{even} \end{cases}$$

$$\lambda(b_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(k_i) = \begin{cases} 1, & i = 1, n \\ 1, & 2 \leq i \leq n-1, \text{even} \\ 2, & 3 \leq i \leq n-1, \text{odd} \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(l_i) = \begin{cases} 1, & 1 \leq i \leq 2n-2, \text{odd} \\ 2, & 2 \leq i \leq 2n-2, \text{even} \end{cases}$$

$$\mu(a_i) = \begin{cases} 3, & 2 \leq i \leq 2n-2, \text{even} \\ 4, & 1 \leq i \leq 2n-2, \text{odd} \end{cases}$$

$$\mu(b_i) = \begin{cases} 3, & i = 1 \\ 4, & i = n \\ 6, & 2 \leq i \leq n-1, \text{even} \\ 7, & 3 \leq i \leq n-1, \text{even} \end{cases}$$

$$\mu(k_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

We see that the sum of adjacent vertices are not identical. Hence, quadrilateral snake graph with pendent edges $PQS_n,\ n \geq 1$ is lucky labeled graph with $\eta(PQS_n) = 2$. □

Theorem 3.8. *Alternate quadrilateral snake graph with pendent edges $PAQS_n$, $n \geq 4$ (even) is lucky labeled graph with $\eta(PAQS_n) = 2$.*

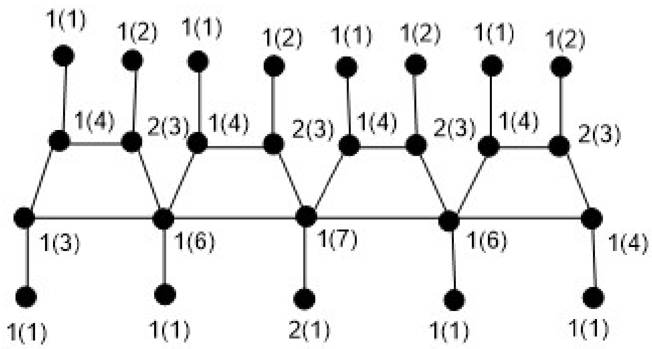

Figure 13: Quadrilateral Snake Graph with Pendent Edges PQS_5

Proof. [11] Alternate quadrilateral snake graph with pendent edges $PAQS_n, n \geq 4$ (even) can be constructed by vertex set $V(PAQS_n) = \{a_i, b_i, l_i, k_i; 1 \leq i \leq n\}$ and edge set $E(PAQS_n) = \{a_i l_i; 1 \leq i \leq n\} \cup \{a_i b_i; 1 \leq i \leq n\} \cup \{b_i k_i; 1 \leq i \leq n\} \cup \{a_{2i-1} a_{2i}; 1 \leq i \leq n/2\} \cup \{b_i b_{i+1}; 1 \leq i \leq n-1\}$. As, the highest vertex degree of the alternate quadrilateral snake graph with pendent edges $PAQS_n$ is 4. So, by theorem 2.7 $\chi(PAQS_n) \leq 5$. If we assign less than two colours then adjacent vertices can't have different colours. By assigning exactly two colours adjacent vertices have different colours. Hence, $\chi(PAQS_n) = 2$. So, by theorem 2.8 we found upper bound $\eta(PAQS_n) \leq 2$. Now we define vertex labeling

$$\lambda : V(PAQS_n) \to \{1, 2\}$$

such that

$$\lambda(l_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(a_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\lambda(b_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(k_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \\ 2, & i = n \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(l_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \text{odd} \\ 2, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\mu(a_i) = \begin{cases} 3, & 2 \leq i \leq n, \text{even} \\ 4, & 1 \leq i \leq n-1, \text{odd} \end{cases}$$

$$\mu(b_i) = \begin{cases} 3, & i = 1 \\ 4, & 3 \leq i \leq n-1, \text{odd} \\ 5, & 2 \leq i \leq n, \text{even} \end{cases}$$

$$\mu(k_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

We see that the sum of adjacent vertices are not identical.
Hence, Alternate quadrilateral snake graph with pendent edges $PAQS_n$, $n \geq 4$ (even) is lucky labeled graph with $\eta(PAQS_n) = 2$. □

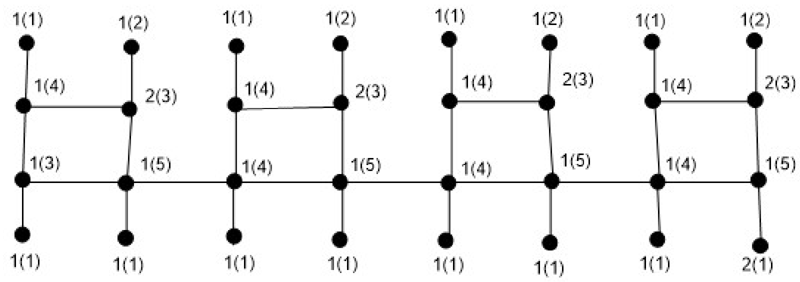

Figure 14: Alternate Quadrilateral Snake Graph with Pendent Edges $PAQS_8$

Theorem 3.9. *Alternate quadrilateral snake graph with pendent edges $PAQS_n$, $n \geq 3$ (odd) is lucky labeled graph with $\eta(PAQS_n) = 2$.*

Proof. [11] Alternate quadrilateral snake graph with pendent edges $PAQS_n$, $n \geq 3$ (odd) can be constructed by vertex set $V(PAQS_n) = \{b_i, k_i; 1 \leq i \leq n\} \cup \{a_i, l_i; 1 \leq i \leq n-1\}$ and edge set $E(PAQS_n) = \{a_i l_i; 1 \leq i \leq n-1\} \cup \{a_i b_i; 1 \leq i \leq n-1\} \cup \{b_i k_i; 1 \leq i \leq n\} \cup \{a_{2i-1}a_{2i}; 1 \leq i \leq (n-1)/2\} \cup \{b_i b_{i+1}; 1 \leq i \leq n-1\}$. As, the highest vertex degree of the alternate quadrilateral snake graph with pendent edges $PAQS_n$ is 4. So, by theorem 2.7 $\chi(PAQS_n) \leq 5$. If we assign less than two colours then adjacent vertices can't have different colours. By assigning exactly two colours adjacent vertices have different colours. Hence, $\chi(PAQS_n) = 2$. So, by theorem 2.8 we found upper bound $\eta(PAQS_n) \leq 2$. Now we define vertex labeling

$$\lambda : V(PAQS_n) \rightarrow \{1, 2\}$$

such that for $n \geq 3$ (odd),

$$\lambda(l_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \end{cases}$$

$$\lambda(a_i) = \begin{cases} 1, & 1 \leq i \leq n-2, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\lambda(b_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

$$\lambda(k_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

Now we evaluate sum of adjacent vertices denoted by μ as follow,

$$\mu(l_i) = \begin{cases} 1, & 1 \leq i \leq n-2, \text{odd} \\ 2, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\mu(a_i) = \begin{cases} 3, & 2 \leq i \leq n-1, \text{even} \\ 4, & 1 \leq i \leq n-2, \text{odd} \end{cases}$$

$$\mu(b_i) = \begin{cases} 3, & i = 1 \\ 2, & i = n \\ 4, & 3 \leq i \leq n-2, \text{odd} \\ 5, & 2 \leq i \leq n-1, \text{even} \end{cases}$$

$$\mu(k_i) = \begin{cases} 1, & 1 \leq i \leq n \end{cases}$$

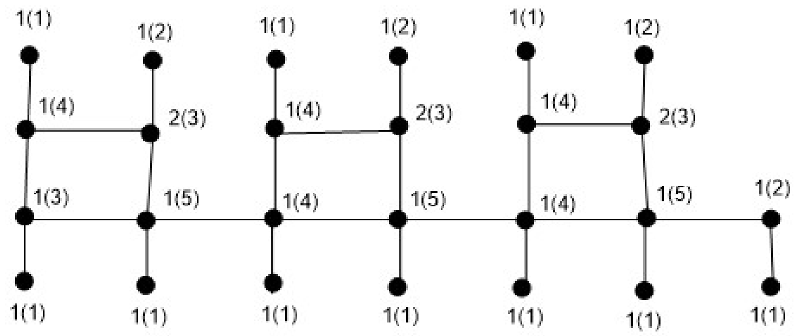

Figure 15: Alternate Quadrilateral Snake Graph with Pendent Edges $PAQS_7$

We see that the sum of adjacent vertices are not identical.
Hence, Alternate quadrilateral snake graph with pendent edges $PAQS_n$, $n \geq 3$ (odd) is lucky labeled graph with $\eta(PAQS_n) = 2$. □

4. conclusion

In this paper, we investigated lucky labelings for several structured families of planar graphs, namely pentagonal snake graphs, alternate pentagonal snake graphs, double pentagonal snake graphs, pentagonal book graphs, hairy cycle graphs, Jahangir graphs, quadrilateral snake graphs with pendant edges, and alternate quadrilateral snake graphs with pendant edges. For each family, an explicit vertex-labeling pattern was presented and the corresponding neighborhood sums were computed to verify that adjacent vertices receive distinct induced values.

The results show that the considered graph families admit lucky labelings with a small number of labels. In particular, the constructed labelings use between 2 and 4 labels and, except for certain parity-dependent cases, the required labeling patterns remain independent of the order of the graph. Thus, despite the increasing size and structural complexity of these families, their lucky-labeling requirements remain uniformly bounded within the constructions presented here. The constructions also indicate that parity, the arrangement of repeated polygonal units, and the presence of pendant vertices are the principal structural features governing the labeling patterns.

This study extends the collection of planar graph families for which explicit lucky labelings are available. Several directions remain open for further investigation. The labeling schemes developed here may be generalized to broader classes of polygonal snake graphs, cactus and unicyclic graphs, graph products, corona-type constructions, and networks obtained by attaching pendant paths or trees. It would also be valuable to determine whether comparable bounds hold for list-additive colorings, proper lucky labelings, and related neighborhood-sum parameters. Finally, computational methods could be developed to test the optimality of the proposed labelings for larger instances and to identify structural conditions under which the lucky number is independent of graph order.

Conflict of Interest: Authors declare no conflict of interest.

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